
Abstract: We consider nonlinear half-wave equations with focusing power-type nonlinearity

\[ i \partial_t u = \sqrt{-\Delta} u - |u|^{p-1}u, \quad \text{with } (t, x) \in \mathbb{R} \times \mathbb{R}^d \]

with exponents \( 1 < p < \infty \) for \( d = 1 \) and \( 1 < p < (d + 1)/(d - 1) \) for \( d \geq 2 \). We study traveling solitary waves of the form

\[ u(t, x) = e^{i\omega t} Q_v(x - vt) \]

with frequency \( \omega \in \mathbb{R} \), velocity \( v \in \mathbb{R}^d \), and some finite-energy profile \( Q_v \in H^{1/2}(\mathbb{R}^d) \), \( Q_v \neq 0 \). We prove that traveling solitary waves for speeds \( |v| \geq 1 \) do not exist. Furthermore, we generalize the non-existence result to the square root Klein–Gordon operator \( \sqrt{-\Delta + m^2} \) and other nonlinearities.

As a second main result, we show that small data scattering fails to hold for the focusing half-wave equation in any space dimension. The proof is based on the existence and properties of traveling solitary waves for speeds \( |v| < 1 \). Finally, we discuss the energy-critical case when \( p = (d + 1)/(d - 1) \) in dimensions \( d \geq 2 \).