1 The Monge-Kantorovich-Vaserstein distance

Let $\Omega$ be a metric space, $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ subsets of $n$ points in $\Omega$. We want to define a distance between $x$ and $y$ which will be meant to be the “work” done to move $x$ into $y$. A possibility is to move $x_i$ into $y_{j(i)}$, $j(i)$ a permutation of $1, \ldots, n$, and summing:

$$\sum_{i=1}^{n} d(x_i, y_{j(i)}) = \sum_{i,j} Q(i,j)d(x_i, y_j)$$

where $Q(i,j) = 1$ if $j = j(i)$ and $0$ otherwise. The choice of the permutation $j(i)$ and hence of $Q$ is arbitrary, it is then natural to define

$$\text{dist}(x, y) := \min_{Q} \sum_{i,j} Q(i,j)d(x_i, y_j) \quad (1.1)$$

We may interpret the above as “mass transport” thinking that all points $x_i$ and $y_i$ have the same mass $m$, thus $d(x_i, y_j)$ in (1.1) is the cost of moving a mass $m$ from $x_i$ to $y_j$. We suppose $m = 1/n$ so that the total masses of $x$ and $y$ are equal to 1 and can be thought of as the support of probability measures $\mu$ and $\nu$ concentrated on the points $x$ and $y$ respectively. This opens the way to several generalizations.

Let $\Omega$ be the Euclidean space $\mathbb{R}^d$ and $\mu$ and $\nu$ probability measures on $\Omega$, the Monge-Kantorovich distance between $\mu$ and $\nu$ is then the generalization of (1.1) to this case where the role of $Q$ is played by transformations $T$ which are isomorphisms between $\mu$ and $\nu$. 

Notes of a course on
Mass transport, Markov chains and Gibbs measures

E. Presutti, GSSI

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In continuum mechanics one studies evolution equations in $\mathbb{R}^d$ determined by the Cauchy problem:

$$\frac{dx(t)}{dt} = v(x(t)), \quad x(0) = x$$

$v(x)$ being the velocity field. We suppose that the solution is $T_t(x)$ and that $T_t$ is for any $t \geq 0$ a smooth, invertible transformation of $\mathbb{R}^d$ onto itself. Then given a probability $\mu$ which represents the initial mass distribution on $\mathbb{R}^d$ we define the mass distribution at time $t$ as

$$\mu_t[B] = \mu[T_t^{-1}B], \quad \text{for any bounded Borel set } B$$

A coupling between $\mu$ and $\nu$ is often searched by looking at the inverse problem: namely find the evolution equation (1.2) such that at a time $\tau$ $\mu_\tau = \nu$. The “right” evolution equation will then produce the “right” coupling. Classical mechanics is an example with the Euler-Lagrange formulation of the Newton equations where in alternative to solving the differential equations of motion one solves the variational problem involving the Lagrangian action. There is a large literature about relations between differential equations and variational problems which is still very active, see for instance the works by Ambrosio and Savaré in Italy, Felix Otto in Germany and Villani in France.

**Exercise 1.1.** Let $\rho(x)$ be the density of $\mu$, call $\rho_t(x)$ the density of $\mu_t$. Is it always true that $\rho_t(x) = \rho(T_t^{-1}x)$? If not, when does it happen? Does it hold when (1.2) is the Hamilton equation of motion?

In the sequel we will study applications to probability and statistical mechanics and consider finite sets $\Omega$ with $\mu$ and $\nu$ probability measures on $\Omega$. In general it will not be possible to move the whole mass $\mu(x)$ into a point $y$ because it may be that all $\nu(y)$ are smaller than $\mu(x)$. We will therefore split the mass $\mu(x)$ into parts called $Q(x,y)$ and move the mass $Q(x,y)$ from $x$ to $y$, the matrix $Q(x,y)$ must then satisfy

$$Q(x,y) \geq 0, \quad \sum_y Q(x,y) = \mu(x), \quad \sum_x Q(x,y) = \nu(y)$$

Such matrices are called “couplings” of the probabilities $\mu$ and $\nu$. The Monge-Kantorovich-Vaserstein distance $D(\mu,\nu)$ between $\mu$ and $\nu$ is then

$$D(\mu,\nu) := \inf_Q \sum_{x,y} Q(x,y)d(x,y)$$

where the inf is now over all couplings of $\mu$ and $\nu$. We will find good couplings $Q$ by studying evolution equations which transport $\mu$ onto $\nu$ after some time $\tau$, similarly to what outlined in the case of continuum mechanics, but here the evolution equations will be stochastic.
2 Applications to spin systems.

An Ising spin configuration $\sigma_\Lambda$ on $\Lambda$, $\Lambda$ a finite subset of $\mathbb{Z}^d$, $d \geq 1$, is a function $\sigma(x), x \in \Lambda$, with values on $\{-1, 1\}$, thus $\sigma_\Lambda$ is an element of $\{-1, 1\}^\Lambda$. More generally a spin configuration $s$ is a function $s_x, x \in \Lambda$, with values in a finite set $S$, thus $s \in \Omega = S^\Lambda$ and for a while we will consider a setup where $s$ belongs to a finite set $\Omega$ not necessarily requiring that $\Omega = S^\Lambda$.

Let $d(s, s')$ be a distance on $\Omega$, $\mu$ and $\nu$ probability distributions on $\Omega$, we then define, see (1.5),

$$ D(\mu, \nu) := \inf_{Q} \sum_{s, s'} Q_{s, s'} d(s, s') $$

where the inf is over all the couplings between $\mu$ and $\nu$. The set of couplings is non empty, it contains for instance the “trivial coupling” $Q(s, s') = \mu(s)\nu(s')$. A more interesting one is obtained by putting as much mass as possible on the diagonal:

$$ Q(s, s) := \alpha_s = \min \{\mu(s), \nu(s)\}, \quad M := \sum_{s \in \Omega} \alpha_s $$

$$ Q_{s, s'} := \alpha_s 1_{s = s'} + \frac{1}{1 - M} [\mu(s) - \alpha_s][\nu(s') - \alpha_{s'}] $$

Prove as an exercise that this is indeed a coupling.

Recall that the total variation distance between $\mu$ and $\nu$ is:

$$ \|\mu - \nu\|_{tv} = \sum_{s \in S} |\mu(s) - \nu(s)| $$

**Exercise** Prove that

$$ \|\mu - \nu\|_{tv} = 2D(\mu, \nu) $$

where $D(\mu, \nu)$ is given by (2.1) with $d(s_\Lambda, s'_\Lambda) = 1_{s_\Lambda \neq s'_\Lambda}$. Hint: use the coupling (2.2).

In the applications we are often interested in proving only that $D(\mu, \nu)$ is small, to this end it is not necessary to really find the inf in (2.1) but it is enough to have a good coupling $Q$. This may however be a difficult task if $\Lambda$ is large as it happens in many applications. We will discuss the issue in the case of Markov chains and Gibbs measures showing how good couplings can be found.

3 Markov chains.

A Markov chain is characterized by a state space $S$ (which we suppose to be a finite set) and by a “transition probability” $p(s, s')$, i.e. for any $s \in S$, $p(s, \cdot)$ is a probability on $S$. 

The physical interpretation is of a random walk on $S$ which at any integer time jumps from $s$ to $s'$ with probability $p(s, s')$.

Let $\mu_0$ be a probability on $S$ that we interpret as the initial distribution of the random walk, then the probability that at time 1 the walk is at position $s$ is the sum

$$\sum_{s' \in S} \mu_0(s')p(s', s)$$

namely the probability that it is initially at $s'$ and then that it jumps to $s$, we then have to sum over all initial states $s'$. Here is the basic assumption of the Markov chain, namely that the probability of a jump $s \rightarrow s'$ is not affected by the distribution probability before the jump. Given an initial probability $\mu_0$ we then define iteratively for any $n \geq 1$

$$\mu_n(s) := \sum_{s'} \mu_{n-1}(s')p(s', s) \quad (3.1)$$

as the distribution probability of the random walk at time $n$.

**Definition.** $\mu$ is invariant if $\mu_1 = \mu$, i.e.

$$\mu(s) = \sum_{s'} \mu(s')p(s', s) \quad (3.2)$$

and hence $\mu_n = \mu$ for all $n \geq 1$.

**Theorem 3.1.** Suppose that $p(s, s') \geq \delta > 0$ for all $s$ and $s'$, then there is a unique invariant measure $\mu$ and given any two initial distributions $\mu_0$ and $\nu_0$ for all $n \geq 1$

$$\|\mu_n - \nu_n\|_{tv} \leq \|\mu_0 - \nu_0\|_{tv}(1 - \delta)^n \quad (3.3)$$

**Proof.** (3.3) follows by iteration from the case $n = 1$. Uniqueness follows from (3.3) with $n = 1$ because $\mu_1 = \mu_0$ if $\mu_0$ is invariant and therefore $\|\mu_0 - \nu_0\|_{tv} \leq \|\mu_0 - \nu_0\|_{tv}(1 - \delta)$. To prove (3.3) with $n = 1$ we use Exercise 2.1 to write

$$\|\mu_1 - \nu_1\|_{tv} = 2D(\mu_1, \nu_1) \quad (3.4)$$

where $D(\mu, \nu)$ is the Vaserstein distance between $\mu$ and $\nu$ with $1_{s,s'}$ the distance in $S$. Thus

$$\|\mu_1 - \nu_1\|_{tv} \leq 2 \sum_{s,s'} Q(s,s')1_{s \neq s'} \quad (3.5)$$

if $Q$ is a coupling of $\mu_1$ and $\nu_1$: we will not need $Q$ to be the coupling which realizes the total variation distance between $\mu_1$ and $\nu_1$. Given $s, s'$ in $S$ let $q(s,t; s', t')$ be a coupling of $p(s, s')$ and $p(t, t')$; let $Q_0(s, t)$ be a coupling of $\mu_0$ and $\nu_0$ then

**Exercise 3.1.**

$$Q_1(s', t') := \sum_{s_0,t_0} Q_0(s_0, t_0)q(s_0, t_0; s', t') \quad (3.6)$$
is a coupling of $\mu_1$ and $\nu_1$.

Define
\[
p(s, t; s', t') := p(s, s')p(t, t'), \text{if } s \neq t, \quad p(s, t; s', t') := p(s, s')1_{t'=s'}, \text{if } s = t \tag{3.7}
\]

**Exercise 3.2.** Prove that $p(s, t; s', t')$ is a transition probability on the space $S \times S$ and that it is a coupling of $p(s, s')$ and $p(t, t')$.

We then choose $Q_1(s', t')$ in (3.6) by taking $Q_0(s_0, t_0)$ as the coupling (2.2) (where as much mass as possible is put on the diagonal) and letting $q(s_0, t_0; s', t')$ be the coupling $p(s_0, t_0; s', t')$ defined in (3.7). We leave it as an exercise to prove that
\[
\frac{1}{2}\|\mu_1 - \nu_1\|_{tv} \leq \sum_{s', t'} Q_1(s', t')1_{s'=s'} \leq (1 - \delta) \sum_{s, t} Q_0(s, t)1_{t\neq s} = (1 - \delta) \frac{1}{2}\|\mu_0 - \nu_0\|_{tv} \tag{3.8}
\]
which completes the proof of the theorem. \hfill \Box

**Exercise 3.3.** Find the invariant probability $\nu(s', t')$ for the Markov chain in $S \times S$ with transition probability $p(s, t; s', t')$.

**Exercise 3.4.** Let $p(s, s')$ be a transition probability which does not necessarily satisfy the assumption in Theorem 3.1. Let $\mu_0$ be a probability on $S$ and $\mu_i$ the corresponding probability at time $i$. Prove that the Cesaro average
\[
\nu_N := \frac{1}{N+1} \sum_{i=0}^{N} \mu_i \tag{3.9}
\]
converges by subsequences to a measure $\mu$ which is invariant, i.e. such that (3.2) holds.

**Exercise 3.5.** Prove the analogue of Theorem 3.1 when the assumption $p(s, s') \geq \delta > 0$ for all $s$ and $s'$ is replaced by
\[
p^2(s, s') := \sum_{t \in S} p(s, t)p(t, s') \geq \delta > 0, \quad \text{for all } s, s' \in S \tag{3.10}
\]

The above has a nice interpretation in terms of Hilbert spaces. Define the scalar product
\[
\langle f, g \rangle = \sum_s f(s)g(s) \tag{3.11}
\]
and denote by $p$ the operator
\[
\langle f, pg \rangle = \sum_s f(s) \sum_{s'} p(s, s')g(s') \tag{3.12}
\]
Its adjoint \( p^+ \) is the operator with kernel \( p^+(s, s') = p(s', s) \). Then \( \mu \) is invariant under \( p \) iff \( \mu \) is an eigenvector of \( p^+ \) with eigenvalue 1 and we have

\[
\langle \mu, pg \rangle = \langle p^+ \mu, g \rangle = \langle \mu, g \rangle
\]  

(3.13)

Suppose \( \mu \) invariant and \( \mu(s) > 0 \) for all \( s \), we can then define the new scalar product

\[
\langle f, g \rangle_\mu = \langle \mu f, g \rangle
\]  

(3.14)

**Definition.** A probability \( \mu \) on \( S \) has the “detailed balance” property with respect to the transition probability \( p(s, s') \) if

\[
\mu(s)p(s, s') = \mu(s')p(s', s) \quad \text{for all } s, s' \in S
\]  

(3.15)

Prove as an exercise that if (3.15) holds then \( \mu \) is invariant.

**Exercise 3.6.** Let \( \mu \) be invariant. Compute the adjoint \( p^* \) of \( p \) in the space with scalar product \( \langle f, g \rangle_\mu \) and prove that the detailed balance condition is equivalent to requiring that \( p \) is self-adjoint.

A Markov chain with “transition” probability \( p(s, s') \) and initial distribution \( \mu_0 \) is the probability \( P_{\mu_0, p} \) on \( S^\mathbb{N} \) defined by setting

\[
P_{\mu_0, p} \left[ s_0 = a_0, \ldots, s_k = a_k \right] = \mu_0(a_0)p(a_0, a_1) \cdots p(a_{k-1}, a_k)
\]  

(3.16)

**Exercise 3.7.** Use theorems in measure theory to prove that \( P_{\mu_0, p} \) defined by (3.16) (for all \( k \) and all \( a_0, \ldots, a_k \)) extends uniquely to a probability on \( S^\mathbb{N} \).

**Exercise 3.8.** (Weak law of large numbers). Let \( f : S \to \mathbb{R} \) and let \( \mu \) and \( p(s, s') \) as in Theorem 3.1. Then for any \( \delta > 0 \)

\[
\lim_{k \to \infty} P_{\mu_0, p} \left[ \left| \frac{1}{k+1} \sum_{i=0}^{k} f(s_i) - \langle f \rangle_\mu \right| > \delta \right] = 0, \quad \langle f \rangle_\mu = \sum_{s \in S} f(s)\mu(s)
\]  

(3.17)

Let \( \mu \) be invariant for \( p(s, s') \), then the law of the Markov chain on \( S^\mathbb{Z} \) is defined by setting

\[
\mathbb{P}_{\mu, p} \left[ s_{-k} = a_{-k}, \ldots, s_k = a_k \right] = \mu(a_{-k})p(a_{-k}, a_{-k+1}) \cdots p(a_{k-1}, a_k)
\]  

(3.18)

**Exercise 3.9.** Use theorems in measure theory to prove that \( \mathbb{P}_{\mu, p} \) defined by (3.18) (for all \( k \) and all \( a_{-k}, \ldots, a_k \)) extends uniquely to a probability on \( S^\mathbb{Z} \). Where do we use the assumption that \( \mu \) is invariant ?

**Exercise 3.10.** (Time reversal). What is the relation between \( \mathbb{P}_{\mu, p} \) and \( \mathbb{P}_{\mu, p^*} \), where \( p^* \) is the adjoint of \( p \) in the space with scalar product \( \langle f, g \rangle_\mu \).
4 Large deviations and the transfer matrix.

We have seen in Exercise 3.9 that if $f : S \rightarrow \mathbb{R}$ and moreover $\mu$ and $p(s, s')$ are as in Theorem 3.1 then (3.17) holds, namely for any $\delta > 0$:

$$
\lim_{k \to \infty} P_{\mu_0, p} \left[ \left| \frac{1}{k+1} \sum_{i=0}^{k} f(s_i) - \langle f \rangle_\mu \right| > \delta \right] = 0, \quad \langle f \rangle_\mu = \sum_{s \in S} f(s) \mu(s)
$$

Let

$$
\phi \in (\min f, \max f), \quad \phi \neq \langle f \rangle_\mu
$$

Then

$$
\lim_{\delta \to 0} \lim_{k \to \infty} P_{\mu_0, p} \left[ \left| \frac{1}{k+1} \sum_{i=0}^{k} f(s_i) - \phi \right| \leq \delta \right] = 0
$$

The large deviation problem refers to the rate of convergence in the above limit, namely to estimate

$$
\lim_{\delta \to 0} \lim_{k \to \infty} \frac{1}{k+1} \log P_{\mu_0, p} \left[ \left| \frac{1}{k+1} \sum_{i=0}^{k} f(s_i) - \phi \right| \leq \delta \right]
$$

The idea due to Cramer is to change $P_{\mu_0, p}$ into a new measure that we denote by $P_{\theta, \mu_0, p}$ so that for this (i) the law of large numbers holds and (ii) the $P_{\theta, \mu_0, p}$ average of $f$ is $\phi$. We will then need to estimate the Radon-Nikodim derivative $P_{\mu_0, p}(s) / P_{\theta, \mu_0, p}(s)$. It is beyond the purposes of this course to carry out the above program and we will just do a few steps which will be then used in the analysis of the one dimensional Gibbs measure.

Calling

$$
A_{k, \delta}(s) := \left| \frac{1}{k+1} \sum_{i=0}^{k} f(s_i) - \phi \right| \leq \delta
$$

we write

$$
P_{\mu_0, p}[A_{k, \delta}] = \sum_s 1_{A_{k, \delta}}(s) P_{\mu_0, p}(s) e^{\theta(\sum_{i=0}^{k} f(s_i))} e^{-\theta(\sum_{i=0}^{k} f(s_i))}
$$

and by an abuse of notation

$$
P_{\mu_0, p}[A_{k, \delta}] = e^{-\theta(k+1)\phi \pm \theta(k+1)\delta} \sum_s 1_{A_{k, \delta}}(s) P_{\mu_0, p}(s) e^{\theta(\sum_{i=0}^{k} f(s_i))}
$$

which becomes

$$
P_{\mu_0, p}[A_{k, \delta}] = e^{-\theta(k+1)\phi \pm \theta(k+1)\delta} Z_k \theta P_{\theta, \mu_0, p}[A_{k, \delta}]
$$

where

$$
Z_k = \sum_s P_{\mu_0, p}(s) e^{\theta(\sum_{i=0}^{k} f(s_i))}
$$

$$
P_{\mu_0, p}(s) = \frac{1}{Z_k} Z_k P_{\mu_0, p}(s) e^{\theta(\sum_{i=0}^{k} f(s_i))}
$$
So one of the steps we have to do is to estimate $Z_θ^k$ and then to prove a weak law of large numbers for $P^θ_{µ_0,p}$. All that is done by reducing $P^θ_{µ_0,p}$ to the law of a Markov chain with a new transition probability. As already mentioned the other steps to prove the large deviation estimate will not be done here. We just observe that

$$\lim_{k \to \infty} \frac{1}{k+1} \log P^θ_{µ_0,p}[A_{k,δ}] \leq \inf_θ \left( -θ(k+1)φ + \lim_{k \to \infty} \frac{1}{k+1} \log Z_θ^k \right)$$

so that the above problem can be rephrased in terms of a variational problem.

For later purposes it is convenient to phrase the problem as follows. Let $Λ$ be an interval $[-N, N]$, $N \in \mathbb{N}$, $s = (s_{-N}, ..., s_N)$ a spin configurations in $S[-N,N]$, $S$ a finite set. The process is characterized by an initial and a final state, respectively $s_{-N-1} ∈ S$ and $s_{N+1} ∈ S$ and by a matrix (the transfer matrix) $M(s, s') ≥ 0, s, s' ∈ S$. In the above large deviations problem $M(s, s') = p(s, s')e^{θf(s)}$

We replace (3.16) by

$$P_{s_{-N-1}, s_{N+1}}(s) = \frac{1}{Z_{s_{-N-1}, s_{N+1}}^{-1}} \prod_{i=-N-1}^{N} M(s_i, s_{i+1})$$

(4.1)

where the partition function $Z_{s_{-N-1}, s_{N+1}}$ is

$$Z_{s_{-N-1}, s_{N+1}} = \sum_s \prod_{i=-N-1}^{N} M(s_i, s_{i+1})$$

(4.2)

so that $P_{s_{-N-1}, s_{N+1}}(s)$ is a probability. We are again interested in the dependence on the boundary conditions which are here $s_{-N-1}$ and $s_{N+1}$ and in particular we want to look for instance at the distribution of $s_0$ which is away from the boundaries. Our purpose therefore is to show that $D(µ_0^N, ν_0^N)$ vanishes where

$$µ_0^N(s) := P_{s_{-N-1}, s_{N+1}}[s_0 = s], \quad ν_0^N(s) := P_{t_{-N-1}, t_{N+1}}[s_0 = s]$$

(4.3)

The Perron-Frobenius theorem allows to state the problem in the context of Markov chains as we are going to see.

**Theorem 4.1. (Perron-Frobenius)** Suppose that $M(s, s') ≥ δ > 0$ for all $s, s' ∈ S$. Then $M$ has a strictly positive eigenvalue $λ$ with eigenvector $ψ$ whose components are all strictly positive. Moreover the rest of the spectrum of $M$ is in an interval $[-λ + a, λ - a]$ with $a > 0$.

**Exercise 4.1.** Suppose further that $M$ is a transition probability, find $λ$ and $ψ$ and a bound for $a$. 

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As a consequence
\[ p(s, s') := \frac{\psi(s')}{\lambda \psi(s)} M(s, s') \] (4.4)
is a transition probability and the numerator in (4.1) can be written as
\[ \frac{\lambda^{2N+2} \psi(s_{-N-1})}{\psi(s_{N+1})} \prod_{i=-N-1}^{N} p(s_i, s_{i+1}) \] (4.5)
Thus \( \mu_0^N(s) \) defined in (4.3) is equal to
\[ \mu_0^N(s) = \left[ \frac{\lambda^{2N+2} \psi(s_{-N-1})}{\psi(s_{N+1})} \prod_{i=-N-1}^{N} p(s_i, s_{i+1}) \right]^{-1} p^{N+1}(s_{-N-1}, s)p^{N+1}(s, s_{N+1}) \] (4.6)
Call \( \mu^\text{inv}(s) \) the invariant measure for the Markov chain with transition probability \( p(s, s') \), then \( \mu_0^N(s) \to \mu^\text{inv}(s) \) exponentially fast:
\[ |\mu_0^N(s) - \mu^\text{inv}(s)| \leq ce^{-bN}, \quad c, b > 0 \] (4.7)
Analogously
\[ |\nu_0^N(s) - \mu^\text{inv}(s)| \leq ce^{-bN}, \quad c, b > 0 \] (4.8)
so that, by (2.4),
\[ D(\mu_0^N, \nu_0^N) = \frac{1}{2} \sum_{s \in S} |\mu_0^N(s) - \nu_0^N(s)| \leq \frac{1}{2} |S|2ce^{-bN} \] (4.9)
hence the desired exponential convergence.

5 Gibbs measures and the Ising model.

Roughly speaking the basic axioms of Statistical Mechanics are:

- The state of a system at equilibrium with inverse temperature \( \beta \) is proportional to \( e^{-\beta H} \), \( H \) the energy of the state. The normalization factor \( Z_\beta \) which makes it a probability is called “the partition function”.

- The thermodynamic pressure is given by \( 1/(\beta|\Lambda|) \) times the log of the partition function (in the limit when \( \Lambda \) invades the whole space).

- The above should be complemented by a theorem which states that the limit exists and that the equation of state derived in this way is “compatible” with thermodynamics.
To explain the meaning of the above definition we examine the particular case of the Ising model which is a very schematic representation of a magnet. The Ising model is a lattice system, at each site $x$ of the lattice there is a magnetic moment, spin, $\sigma(x)$: we suppose that the magnetic moment has fixed modulus (say 1) and it can only be oriented up or down, thus $\sigma(x) = \pm 1$. Suppose that the system is confined to a finite region $\Lambda \subset \mathbb{Z}^d$ so that the state of the system is specified by the spin configurations $\sigma_\Lambda = \{\sigma(x), x \in \Lambda\} \in \{-1, 1\}^\Lambda$. The energy of a configuration $\sigma_\Lambda$ is

$$H_\Lambda(\sigma_\Lambda) = -\frac{1}{2} \sum_{x \sim y, x, y \in \Lambda} J \sigma(x) \sigma(y)$$

(5.1)

where $x \sim y$ means that $x$ and $y$ are nearest neighbor in $\mathbb{Z}^d$. $J$ is the coupling constant, we suppose in the sequel that $J > 0$ which means that the interaction energy between two spins is minimal when they are aligned, this is why the case $J > 0$ is referred to as ferromagnetic. More general hamiltonians are studied: longer range, not necessarily ferromagnetic interactions. We will also consider generalizations of (5.1) setting

$$H_{\Lambda,h}(\sigma_\Lambda|\bar{\sigma}_\Lambda) = H_\Lambda(\sigma_\Lambda) - h \sum_{x \in \Lambda} \sigma(x) - \sum_{x \sim y, x, y \in \Lambda, y \notin \Lambda} \sigma(x) \bar{\sigma}(y)$$

(5.2)

Here $\bar{\sigma}_\Lambda$ represents a spin configuration outside $\Lambda$ and the last term in (5.2) gives the interaction energy between the spins in $\Lambda$ and those outside. $h$ is an external magnetic field, its interaction energy with a spin $\sigma(x)$ is $-h\sigma(x)$, thus its effect is to make the spins have the same sign as $h$ because in such a case the energy is minimal.

Given a parameter $\beta > 0$ the Gibbs measure in $\Lambda$ is

$$\mu_{\Lambda,\beta,h,\bar{\sigma}_\Lambda}(\sigma_\Lambda) := \frac{1}{Z_{\Lambda,\beta,h,\bar{\sigma}_\Lambda}} e^{-\beta H_{\Lambda,h}(\sigma_\Lambda|\bar{\sigma}_\Lambda)}$$

(5.3)

where

$$Z_{\Lambda,\beta,h,\bar{\sigma}_\Lambda} = \sum_{\sigma_\Lambda} e^{-\beta H_{\Lambda,h}(\sigma_\Lambda|\bar{\sigma}_\Lambda)}$$

(5.4)

(we may drop $\bar{\sigma}_\Lambda$ and $h$ when they are absent). Thermodynamics is defined by taking $T = 1/(k\beta)$ as the absolute temperature and defining the thermodynamical pressure as

$$P(\beta, h) := \lim_{\Lambda \to \mathbb{Z}^d} \frac{1}{|\Lambda|} \log Z_{\Lambda,\beta,h,\bar{\sigma}_\Lambda}$$

(5.5)

where $\Lambda$ is a sequence of increasing regions which invades the whole space (conditions on the sequence will be specified later); moreover the limit is independent of the sequence $\bar{\sigma}_\Lambda$. Thus all that requires theorems (proven in much more generality) which state that the above conditions are satisfied and that the function $P(\beta, h)$ has the right convexity properties which the thermodynamic pressure should have.

In my opinion the main justification to the above axioms comes from the good agreement with the experiments and because it provides a link between the macroscopic behavior and the microscopic interactions, that is a base for molecular engineering.
From a mathematical viewpoint however one is more ambitious and would like to derive the axioms from more fundamental theories. One attempt is via the ergodic theory where the purpose is to derive Statistical Mechanics from the Hamilton equations. The ergodic assumption is that there is only one invariant measure (absolutely continuous with respect to the area measure) on the surface of constant energy. It is then natural to take this as the equilibrium measure and indeed one can prove that in the limit of large systems such a measure behaves as the Gibbs measure. The ergodic property is however very far from being proven in systems of interest in statistical mechanics.

Another approach is due to Boltzmann who proposed a formula for the entropy which should be proportional to the log of the number of states at given energy $E$, an approach which underlines closeness between statistical mechanics and information theory. There are theorems which state that this approach is consistent with the Gibbs assumption.

The above axiomatic approach does not tell anything about the thermodynamic transformations, Carnot cycle,..., for this one has to investigate the dynamical properties of the system, the knowledge of the invariant measures being not sufficient. Examples where this can be done have been discussed by Stefano Olla.

6 The one dimensional Ising model.

The Gibbs measure in $d = 1$ dimension is very special and simple. Let $\Lambda$ in (5.3) be the interval $[-N,N]$, denoted by $\Lambda_N$. $\bar{\sigma}_N$ is a configuration outside $\Lambda_N$. Notice that the Gibbs measure $\mu_{\Lambda_N,0,h,\bar{\sigma}_N}(\sigma_{\Lambda_N})$ depends only on two spins: $\bar{\sigma}_{\pm(N+1)}$. Its analysis can be reduced to that of Section 4 because the Gibbs measure in (5.3) can be written as:

$$Z_{\Lambda,0,h,\bar{\sigma}_c}^{-1} e^{-\beta h \bar{\sigma}_{-N-1}} \prod_{i=-N-1}^{N} M(\sigma_i, \sigma_{i+1}), \quad M(\sigma_i, \sigma_{i+1}) = e^{\beta J \sigma_i \sigma_{i+1} + h \sigma_i}$$  \hspace{1cm} (6.1)

with the understanding that $\sigma(x) = \bar{\sigma}(x)$ if $x = \pm(N+1)$. Moreover

$$Z_{\Lambda,0,h,\bar{\sigma}_c} = \sum_{\sigma(-N),..,\sigma(N)} e^{-\beta h \bar{\sigma}_{-N-1}} \prod_{i=-N-1}^{N} M(\sigma_i, \sigma_{i+1})$$  \hspace{1cm} (6.2)

We are thus in the same setup of Section 4 (because $M(\sigma, \sigma') \geq \delta > 0$) and therefore the distribution of $\sigma(0)$ becomes independent of $\bar{\sigma}_{\pm(N+1)}$ in the limit $N \to \infty$. Same property for the distribution of $\sigma(-k),..,\sigma(k)$ (left as an exercise). Existence of the pressure then follows because

$$\lim_{N \to \infty} \frac{1}{\beta(2N+1)} \log Z_{\Lambda,0,h,\bar{\sigma}_c} = \frac{\log \lambda}{\beta}$$  \hspace{1cm} (6.3)

where $\lambda$ is the maximal eigenvalue of $M$. 

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The asymptotic independence from the boundary conditions is physically interpreted as “absence of phase transitions”. As we will see this is not always true when \( d \geq 2 \) where phase transitions may occur.

7 Thermodynamic limit for the pressure.

We will prove that:

**Theorem 7.1.** Let \( \Lambda_n \subset \mathbb{Z}^2 \) be a sequence of squares of side \( n \), then

\[
\lim_{n \to \infty} p_{\Lambda_n, \beta, h} \equiv \lim_{n \to \infty} \frac{1}{\beta|\Lambda_n|} \log Z_{\Lambda, \beta, h} =: p_{\beta, h}
\]

where \( Z_{\Lambda, \beta, h} \) is equal to \( Z_{\Lambda_{n, \beta, h}, \sigma_{\Lambda_c}} \) defined in (5.4) with \( \sigma_{\Lambda_c} = 0 \).

Before the proof some remarks: the existence of the thermodynamic limit for the pressure holds under much more general assumptions. The restriction to \( d = 2 \) is only for drawing pictures, the interaction does not need to be nearest neighbor and \( \sigma_{\Lambda_c} \) can be arbitrary. Finally \( \Lambda_n \) may be more general than squares (van Hove sequences are allowed, see the exercises at the end of the section).

**Proof.** There is a constant \( c \) so that for any bounded region \( \Gamma \) and any \( \sigma_{\Gamma_c} \)

\[
|H_{\Gamma, h}(\sigma_{\Gamma_c})| \leq c|\Gamma|
\]

Thus the limit in (7.1) exists by subsequences and \( p_{\beta, h} \leq c \). We will first prove the existence of the limit for a special subsequence \( \Delta_n \) of squares of side \( \ell_n = 2^n \). The square \( \Delta_n \) is the union of four squares \( \Delta_{n-1}(i), i = 1, \ldots, 4 \), of side \( \ell_{n-1} \) and

\[
|H_{\Delta_n, h}(\sigma_{\Delta_n}) - \sum_{i=1}^{4} H_{\Delta_{n-1}(i), h}(\sigma_{\Delta_{n-1}(i)})| \leq 2J\ell_n
\]

Therefore

\[
4 \log Z_{\Delta_{n-1}, \beta, h} - 2J\ell_n \leq \log Z_{\Delta_n, \beta, h} \leq 4 \log Z_{\Delta_{n-1}, \beta, h} + 2J\ell_n
\]

Call \( p_{\beta, h}(\Gamma) = \frac{1}{|\Gamma|} \log Z_{\Gamma, \beta, h} \), divide (7.3) by \( |\Delta_n| \) then

\[
p_{\beta, h}(\Delta_{n-1}) - 2J\ell_n^{-1} \leq p_{\beta, h}(\Delta_n) \leq p_{\beta, h}(\Delta_{n-1}) + 2J\ell_n^{-1}
\]

which proves that \( p_{\beta, h}(\Delta_{n-1}) \) is a Cauchy sequence whose limit is called \( p_{\beta, h} \).

Let us now go back to the original sequence of squares \( \Lambda_n \) (of side \( n \)). We fix \( m \) and given \( \Lambda_n \) we put in \( \Lambda_n \) disjointedly and lexicographically squares \( \Delta_m \) calling \( \Gamma_{n,m} \) the space
in $\Lambda_n$ left empty, thus $|\Gamma_{n,m}| \leq 2n\ell_m$ (why the factor 2?). Calling $k$ the number of squares $\Delta_m$ contained in $\Lambda_n$ and $c$ as in (7.2) we have (left as an exercise)

$$k \log Z_{\Delta_m,\beta,\ell,\ell_m} - \beta c 2n \ell_m - \beta J 2n \frac{n}{\ell_m} \leq \log Z_{\Lambda_n,\beta,\ell,\ell_m} + k \log Z_{\Delta_m,\beta,\ell,\ell_m} + \beta c n \ell_m + \beta J 2n \frac{n}{\ell_m} \quad (7.6)$$

We divide by $\beta |\Lambda_n|$ and get

$$\frac{k |\Delta_m|}{|\Lambda_n|} p_{\beta,\ell,\ell_m}(\Delta_m) - 2cJ \frac{\ell_m}{n} - \frac{J}{\ell_m} \leq p_{\beta,\ell,\ell_m}(\Lambda_n) \leq k |\Delta_m| p_{\beta,\ell,\ell_m}(\Delta_m) + 2cJ \frac{\ell_m}{n} + \frac{J}{\ell_m} \quad (7.7)$$

Since

$$\lim_{n \to \infty} \frac{k |\Delta_m|}{|\Lambda_n|} = 1$$

we get (7.1) by letting first $n \to \infty$ and then $m \to \infty$.

**Exercise 7.1.** Extend (7.1) to the case with boundary conditions, i.e. when $Z_{\Lambda_n,\beta,\ell,\ell_m}$ is replaced by $Z_{\Lambda_n,\beta,\ell,\ell_m,\sigma}$.

**Exercise 7.2.** Prove that $p_{\Lambda_n,\beta,\ell,\ell_m}$ is a convex function of $\beta$ and $h$.

Define the $m$-internal measure of a bounded set $\Gamma \subset \mathbb{Z}^2$ as the max of the area of union of disjoint sets $\Delta_m \subset \Gamma$. The $m$-external measure of $\Gamma$ is the minimal area of the union of disjoint sets $\Delta_m$ covering $\Gamma$. A van Hove sequence $\Gamma_n$ is an increasing sequence of bounded sets $\Gamma_n$ which invades $\mathbb{Z}^2$ and it is such that for any $m$ the ratio of the $m$-internal and the $m$-external measures of $\Gamma_n$ converges to 1 as $n \to \infty$.

**Exercise 7.3.** Extend (7.1) to van Hove sequences.

### 8 The DLR property.

Recall that the invariant measures $\mu(s)$ for Markov chains with transition probability $p(s, s')$ satisfy the invariance condition:

$$\sum_{s'} \mu(s') p(s', s) = \mu(s) \quad (8.1)$$

Also the Gibbs measures enjoy a similar property, called the DLR property. Define

$$p_{x,\beta,\ell,\ell_m}(\sigma, \sigma') := Z_{\{x\},\beta,\ell,\ell_m}^{-1} e^{-\beta H_{\{x\},\beta,\ell,\ell_m}(\sigma_{\{x\}}) - \beta H_{\{x\},\beta,\ell,\ell_m}(\sigma'_{\{x\}}) + \beta H_{\{x\},\beta,\ell,\ell_m}(1_{\sigma_{\{x\}} = \sigma'_{\{x\}}})} \quad (8.2)$$

The right hand side if we neglect the last characteristic function is the Gibbs probability of $\sigma'_{\{x\}}$ in the region consisting of the singleton $\{x\}$ in interaction with $\sigma'_{\{x\}}$. We can regard $p_{x,\beta,\ell,\ell_m}(\sigma, \sigma')$ as a transition probability from $\sigma$ to $\sigma'$: it does not depend on $\sigma_x$ and only the spin at $x$ may change, it is therefore a single spin-flip transformation.
Theorem 8.1. The Gibbs measure \( \mu_{\Delta, \beta, h, \sigma_{\Lambda^c}} \) is invariant under \( p_{x; \beta, h} \) for any \( x \in \Lambda \).

We will prove below a stronger version of Theorem 8.1. Recall that a probability \( \mu(s) \) is invariant under the transition probability \( p(s, s') \) if (8.1) is satisfied. We moreover say that \( \mu \) has the detailed balance property with respect to \( p(s, s') \) if

\[
\mu(s)p(s, s') = \mu(s')p(s', s), \quad \text{for any } s, s'
\]  

(8.3) implies (8.1) (as seen by summing (8.3) over \( s \)). We will prove that

Theorem 8.2. The Gibbs measure \( \mu_{\Delta, \beta, h, \sigma_{\Lambda^c}} \) satisfies detailed balance with respect to \( p_{x; \beta, h} \) for any \( x \in \Lambda \).

Proof. We will use the following equality (whose proof is left as an exercise): for any \( \Delta \subset \Lambda \)

\[
H_{\Lambda, h}(\sigma_{\Lambda} | \bar{\sigma}_{\Lambda^c}) = H_{\Lambda \setminus \Delta, h}(\sigma_{\Lambda \setminus \Delta} | \bar{\sigma}_{\Lambda^c}) + H_{\Delta, h}(\sigma_{\Delta} | \bar{\sigma}_{\Delta^c})
\]  

(8.4) where \( \sigma(x) = \bar{\sigma}(x) \) if \( x \in \Lambda^c \). To simplify notation in the sequel we will drop \( \beta \) and \( h \) from the notation. We use (8.4) with \( \Delta = \{x\} \) and get

\[
\mu(\sigma)p_x(\sigma, \sigma') = Z_{\Lambda, \tau_{\Lambda^c}}^{-1} e^{-\beta H_x(\sigma(x) | \bar{\sigma}_{\Lambda^c})} e^{-\beta H_{\Lambda \setminus \{x\}}(\sigma_{\Lambda \setminus \{x\}} | \bar{\sigma}_{\Lambda^c})} Z_{x, \sigma_{\Lambda^c}'}^{-1} e^{-\beta H_x(\sigma'(x) | \bar{\sigma}_{\Lambda^c})} 1_{\sigma_{\Lambda^c} = \sigma_{\Lambda^c}'}
\]  

(8.5)

Using repeatedly that \( \sigma_{\Lambda^c}' = \sigma_{\Lambda^c} \) the right hand side can be written as

\[
Z_{\Lambda, \tau_{\Lambda^c}}^{-1} e^{-\beta H_x(\sigma'(x) | \bar{\sigma}_{\Lambda^c})} e^{-\beta H_{\Lambda \setminus \{x\}}(\sigma_{\Lambda \setminus \{x\}} | \bar{\sigma}_{\Lambda^c})} Z_{x, \sigma_{\Lambda^c}'}^{-1} e^{-\beta H_x(\sigma(x) | \bar{\sigma}_{\Lambda^c})} 1_{\sigma_{\Lambda^c} = \sigma_{\Lambda^c}'}
\]

which is equal to \( \mu(\sigma')p_x(\sigma', \sigma) \).

\[\square\]

Exercise 8.1. Prove that the conditional probability of \( \mu_{\Lambda, \beta, h, \bar{\sigma}_{\Lambda^c}} \) given \( \sigma_{\Lambda \setminus \{x\}}, x \in \Lambda \), is the Gibbs probability \( \mu_{x; \beta, h, \sigma_{\Lambda^c}} \).

Define for \( \Delta \subset \Lambda \)

\[
p_{\Delta; \beta, h}(\sigma, \sigma') := Z_{\Delta, \beta, h, \sigma_{\Delta^c}'}^{-1} e^{-\beta H_{\Delta, h}(\sigma_{\Delta} | \bar{\sigma}_{\Delta^c})} 1_{\sigma_{\Delta^c} = \sigma_{\Delta^c}'}
\]

(8.6)

Exercise 8.2. Prove that \( p_{\Delta; \beta, h}(\sigma, \sigma') \) is a transition probability and that \( \mu_{\Lambda, \beta, h, \sigma_{\Lambda^c}} \) satisfies detailed balance with respect to \( p_{\Delta; \beta, h}(\sigma, \sigma') \) for any \( \Delta \subset \Lambda \). Prove also that the conditional probability of \( \mu_{\Lambda, \beta, h, \sigma_{\Lambda^c}} \) given \( \sigma_{\Lambda \setminus \Delta}, \Delta \subset \Lambda \), is the Gibbs probability \( \mu_{\Delta, \beta, h, \sigma_{\Delta^c}} \).

We then say that the Gibbs measures satisfy the DLR property.

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9 High temperatures: the Dobrushin theorem.

We will consider in the sequel the Gibbs measure $\mu_{\Lambda_N, \beta, h, \sigma_{\Lambda_N^c}}$ with

$$\Lambda_N = \{ x = (x_1, x_2) \in \mathbb{Z}^2 : |x_1| \leq N, x_2 \leq N \}$$  \hspace{1cm} (9.1)

Fix a bounded set $\Delta$ in $\mathbb{Z}^2$ and take $N$ so large that $\Delta \subset \Lambda_N$. Given any $\sigma^*(x), x \in \Delta$, we want to study the dependence of

$$\mu_{\Lambda_N, \beta, h, \sigma_{\Lambda_N^c}} \left[ \{ \sigma(x) = \sigma^*(x), x \in \Delta \} \right]$$  \hspace{1cm} (9.2)

on $\sigma_{\Lambda_N^c}$ when $N$ is large (and $\Delta$ fixed). We will prove that if $\beta$ is small there is an exponential decay in the distance between $\Delta$ and $\Lambda_N^c$, while if $\beta$ is large and $h = 0$ the dependence persists also in the limit $N \to \infty$, precise statements will be given later.

In (9.2) the spins of $\sigma_{\Lambda_N^c}$ which matter are only those at the boundary of $\Lambda_N$, their number then grows as the surface. It is therefore surprising that they can affect what happens in most of the region $\Lambda_N$ (which goes like the volume) and indeed in general this does not happen. When it does we talk of phase transitions since several measures (phases) may appear by changing the boundary conditions. Notice also that in $d = 1$ we have proved that phase transition cannot occur (for our nearest neighbor interaction).

In this section we will consider small values of $\beta$ (large temperatures) and prove exponential decay. Call

$$\mu_{\Delta, \sigma_{\Lambda_N^c}} (\sigma^*_{\Delta})$$  \hspace{1cm} (9.3)

the quantity in (9.2): it defines a probability $\mu_{\Delta, \sigma_{\Lambda_N^c}}$ on $\{-1, 1\}^\Delta$. Let $\tau_{\Lambda_N^c}$ be another boundary condition, we then want to estimate the Vaserstein distance

$$D' \left( \mu_{\Delta, \sigma_{\Lambda_N^c}}, \mu_{\Delta, \tau_{\Lambda_N^c}} \right) := \inf_{Q_{\Delta}} \sum_{x \in \Delta} |\sigma(x) - \tau(x)| Q_{\Delta}(\sigma, \tau)$$  \hspace{1cm} (9.4)

the inf over all couplings of $\mu_{\Delta, \sigma_{\Lambda_N^c}}$ and $\mu_{\Delta, \tau_{\Lambda_N^c}}$. Notice that $|\sigma(x) - \tau(x)| = 2 1_{\sigma(x) \neq \tau(x)}$, thus, except for the factor 2 this is the distance we used earlier studying the Markov chains.

We write

$$D' \left( \mu_{\Delta, \sigma_{\Lambda_N^c}}, \mu_{\Delta, \tau_{\Lambda_N^c}} \right) \leq D \left( \mu_{\Lambda_N, \sigma_{\Lambda_N^c}}, \mu_{\Lambda_N, \tau_{\Lambda_N^c}} \right) := \inf_{Q_{\Lambda_N}} \sum_{x \in \Delta} |\sigma(x) - \tau(x)| Q_{\Lambda_N}(\sigma, \tau)$$  \hspace{1cm} (9.5)

and we will prove at the end of the section:

**Theorem 9.1.** Let $\beta$ be so small that $\tau := 4\beta J < 1$. Let $n(x, \Lambda^c) + 1$ be the distance of $x$ from $\Lambda_N^c$. Let $\Delta \subset \Lambda_N$. Then

$$D \left( \mu_{\Lambda_N, \sigma_{\Lambda_N^c}}, \mu_{\Lambda_N, \tau_{\Lambda_N^c}} \right) \leq 2 \sum_{x \in \Delta} r^{n(x, \Lambda^c)}$$  \hspace{1cm} (9.6)
The basic ingredient for proving the analogue of Theorem 9.1 for a Markov chain was to find a good coupling for the transition probabilities. Recalling (9.7) the role of the

\[ p \]

Thus we want to couple \( p \) and \( \tau \), and necessarily the form:

\[ P_\sigma = \nu \]

Lemma 9.2. Let \( \mu \) and \( \nu \) be probabilities on \( \{-1, 1\} \) then

\[ D(\mu, \nu) = |E_\mu[\sigma] - E_\nu[\sigma]| \]

which is realized by a coupling \( \pi \) made by a triangular matrix.

Proof. Let \( \mu(1) = p \), \( \nu(1) = p' \) and suppose (without loss of generality) that \( p > p' \). Let \( \pi(\sigma, \sigma') \) be the \( 2 \times 2 \) matrix \( \pi(1, 1), \pi(1, -1), \pi(-1, 1), \pi(-1, -1) \) obtained by putting on the diagonal as mass as possible. Then \( \pi(1, 1) = p' \) and \( \pi(-1, -1) = 1 - p \). To be a coupling the sum on the first row must equal \( \mu(1) = p \), hence

\[ \pi(1, 1) + \pi(1, -1) = \mu(1) = p \]

Analogously the sum on the second row must be equal to \( \mu(-1) = 1 - p \), hence

\[ \pi(-1, 1) + \pi(-1, -1) = \mu(-1) = 1 - p \]

Let us check that \( \pi \) is a coupling. By (9.10) the sum over the second column is \( \pi(-1, -1) + \pi(1, -1) + \pi(1, 1) = 1 - p + (p - p') = 1 - p' = \nu(-1) \) as it should be. The sum over the first column is \( \pi(1, 1) + \pi(-1, 1) = p' = \nu(-1) \) as it should be. Thus \( \pi \) is a coupling of \( \mu \) and \( \nu \) and it is a triangular matrix, as claimed. Moreover

\[ D(\mu, \nu) \leq \sum_{\sigma, \sigma'} \pi(\sigma, \sigma')|\sigma - \sigma'| = 2\pi(1, -1) = |E_\mu[\sigma] - E_\nu[\sigma]| \]

(9.9) then follows because for any coupling \( Q \)

\[ \sum_{\sigma, \sigma'} Q(\sigma, \sigma')|\sigma - \sigma'| \geq |E_\mu[\sigma] - E_\nu[\sigma]| \]

(9.13)
We then choose the coupling $\mathcal{P}$ in (9.8) as

$$q_x(\sigma'', \tau''; \sigma, \tau) = 1_{\sigma(y) = \sigma''(y), \tau(y) = \tau''(y), y \neq x} \pi_{\sigma''_c, \tau''_c}(\sigma(x), \tau(x))$$  \hspace{1cm} (9.14)$$

where $\pi_{\sigma''_c, \tau''_c}(\sigma(x), \tau(x))$ is the coupling in Lemma 9.2. One can check that

$$\sum_{\sigma(x)} Z_{\{x\}, \beta, h, \sigma''(x)c} e^{-\beta H(x, h(\sigma(x))\sigma(x))} \sigma(x) = \tanh \{ \beta [h + J \sum_{|y-x|=1} \sigma''(y)] \}$$

so that by (9.9)

$$\sum_{\sigma(x), \tau(x)} \pi_{\sigma''_c, \tau''_c}(\sigma(x), \tau(x))|\sigma(x) - \tau(x)| = |\tanh t - \tanh t'| \leq |t - t'|$$

$$t = \beta [h + J \sum_{|y-x|=1} \sigma''(y)], \quad t' = \beta [h + J \sum_{|y-x|=1} \tau''(y)]$$  \hspace{1cm} (9.15)$$

Let $\mathcal{P}(\sigma_{\Lambda_N}, \tau_{\Lambda_N})$ be a coupling of $\mu_{\Lambda_N, \sigma_{\Lambda_N}}(\sigma_{\Lambda_N})$ and $\mu_{\Lambda_N, \tau_{\Lambda_N}}(\tau_{\Lambda_N})$, then (proof left as an exercise)

$$\mathcal{P}(\sigma_{\Lambda_N}, \tau_{\Lambda_N}) := \sum_{\sigma_{\Lambda_N}, \tau_{\Lambda_N}} \mathcal{P}(\sigma_{\Lambda_N}, \tau_{\Lambda_N}) q_x(\sigma_{\Lambda_N}, \tau_{\Lambda_N}; \sigma'_{\Lambda_N}, \tau'_{\Lambda_N}) \quad \text{is also a coupling} \quad (9.16)$$

We take

$$\mathcal{P}(\sigma_{\Lambda_N}, \tau_{\Lambda_N}) = \mu_{\Lambda_N, \sigma_{\Lambda_N}}(\sigma_{\Lambda_N}) \mu_{\Lambda_N, \tau_{\Lambda_N}}(\tau_{\Lambda_N})$$  \hspace{1cm} (9.17)$$

which is a coupling but presumably not very good (indeed if $\sigma_{\Lambda_N} = \tau_{\Lambda_N}$ the product measure would not be concentrated on the diagonal). We may hope however that $\mathcal{P}(\sigma'_{\Lambda_N}, \tau'_{\Lambda_N})$ is better than $\mathcal{P}(\sigma'_{\Lambda_N}, \tau'_{\Lambda_N})$ because at least the conditional probabilities at $x$ would be coupled in the optimal way. With this in mind we order the sites in $\Lambda$ as $x_1, \ldots, x_M$ ($M = |\Lambda|$) and define iteratively (using matrix notation)

$$\mathcal{P}^k_j = \mathcal{P'} \left( q_{x_1} \cdots q_{x_M} \right)^{k-1} q_{x_1} \cdots q_{x_j}$$  \hspace{1cm} (9.18)$$

hoping that for $k$ large $\mathcal{P}^k$ is a good coupling, namely, recalling (9.5), that

$$d^k_j(m) := \sum_{\sigma_{\Lambda_N}, \tau_{\Lambda_N}} |\sigma(x_m) - \tau(x_m)| \mathcal{P}^k_j(\sigma_{\Lambda_N}, \tau_{\Lambda_N})$$  \hspace{1cm} (9.19)$$

gets exponentially small when $x_m$ is far away from $\Lambda_N$.

By (9.14)

$$\sum_{\sigma, \tau} q_x(\sigma'', \tau''; \sigma, \tau)|\sigma(y) - \tau(y)| = |\sigma''(y) - \tau''(y)|, \quad y \neq x$$
then
\[ d_j^k(m) = \begin{cases} 
    d_j^k(j) & \text{if } m \geq j \\
    d_j^{k-1}(j) & \text{if } m < j 
\end{cases} \] (9.20)

We are therefore left with the estimate of \( d_j^k(j) \). Let \( x = x_j \) be such that \( y \in \Lambda \) for all \( |y - x| \leq 1 \) then by (9.14)–(9.15)
\[ d_j^k(j) \leq \beta J \sum_{m: |x_m - x_j| = 1} d_{j-1}^k(m), \quad d_0^k(m) := d_{M-1}^k(m) \] (9.21)

Using (9.20) we can iterate (9.21) if the new sites \( x_m \) which appear in (9.21) verify the condition that if \( |y - x_m| = 1 \) then \( y \in \Lambda \). Thus if the dist\((x, \Lambda^c) > k\), calling \( x = x_m \),
\[ d_j^k(m) \leq 2(4 \beta J)^k \] (9.22)

which proves (9.6) and Theorem 9.1.

**Exercise 9.1.** Consider the hamiltonian with spin-spin interaction
\[ -J(x, y)\sigma(x)\sigma(y), \quad \sup_x \sum_{y \neq x} |J(x, y)| < \infty \] (9.23)

Prove that for \( \beta \) small enough there is an analogue of (9.6), in the sense that its left hand side is infinitesimal.

10 Phase transitions at low temperatures.

At zero temperature, \( \beta = \infty \), the ground state of the Ising hamiltonian when \( h > 0 \) is the configuration where all spins are equal to 1, analogously when \( h < 0 \) it is the configuration where all spins are equal to \(-1\). At \( h = 0 \) there are two ground states one made of all +1, the other by all \(-1\). There are two phases, the state with all +1 and the other with all \(-1\): when \( h > 0 \) the +1 phase is selected, the opposite when \( h < 0 \) and at \( h = 0 \) the two phases coexist. When going from \( h > 0 \) to \( h < 0 \) we thus see a phase transition and the line \( h = 0 \) is the coexistence line.

The two dimensional Ising model has a similar structure also when \( T > 0 \ (\beta < \infty) \). Indeed when \( h > 0 \) there is only the + phase where spin configurations are predominantly +1, at \( h < 0 \) they are mostly \(-1\). At \( h = 0 \) things are different. There is a critical inverse temperature \( \beta_c \) and for \( \beta > \beta_c \) there are two measures, one has mostly spins equal to +1, the other equal to \(-1\). For \( \beta \leq \beta_c \) there is only one state where the average magnetization is 0, i.e. configurations have spins with both signs with equal probability.
The Pirogov-Sinai theory shows that for a large class of models the behavior at \( T = 0 \) persists at least for small values of \( T \). In this section we will prove that if \( \beta \) is large enough and \( h = 0 \) for any \( x \)

\[
\lim_{N \to \infty} \mu_{\Lambda_N, \beta, h = 0, \sigma_{\Lambda_N}^c} = \left[ \{ \sigma(x) = 1 \} \right] > \frac{1}{2}
\]

(10.1)

\( \sigma_{\Lambda_N} = \pm \) meaning that all spins in \( \Lambda_N \) are +1, −1, and

\[
\lim_{N \to \infty} \mu_{\Lambda_N, \beta, h = 0, \sigma_{\Lambda_N}^c} = \left[ \{ \sigma(x) = -1, -1 \} \right] > \frac{1}{2}
\]

(10.2)

which shows that the two Gibbs measures are sensitive to the boundary conditions no matter how large is the size of the system. The proof exploits the symmetry at \( h = 0 \) between plus and minus spins which allows to represent spin configurations in terms of contours.

**Contours and related notions.**

- Let \( C_x \) be the closed unit square in \( \mathbb{R}^2 \) with center \( x \in \mathbb{Z}^2 \) and let \( \mathcal{D} \), the dual of \( \mathbb{Z}^2 \), be the union of all the sides of all the squares \( C_x \). The dual lattice of \( \mathbb{Z}^2 \) is instead the union of the corners of all the squares \( C_x \). Two squares \( C_x \) and \( C_y \) are “neighboring squares” if they have a common side (i.e. \( x \) and \( y \) are nearest neighbor sites, \( x \sim y \)).

- Given a spin configuration \( \sigma \) we call disagreement segment a unit segment which is in the intersection of two neighboring squares \( C_x \) and \( C_y \) which have \( \sigma(x) \neq \sigma(y) \). Observe that the disagreement segments are the same for a configuration \( \sigma \) and its spin flip \( -\sigma \), for this reason we will restrict (without loss of generality) to plus boundary conditions, namely to configurations \( \sigma \) in \( \mathbb{Z}^2 \) such that \( \sigma(y) = 1 \) for all \( y \in \Lambda_N^c \). We leave as an exercise that:

- Each point of the dual lattice may only belong to an even number of disagreement segments, i.e. 0, 2, 4.

- All disagreement segments are contained in \( \mathcal{D}_{N+\frac{1}{2}} \) where \( \mathcal{D}_{N+\frac{1}{2}} \) is the intersection of \( \mathcal{D} \) with the square in \( \mathbb{R}^2 \) of side \( 2[N + \frac{1}{2}] + 1 \) and center the origin.

- The disagreement set \( \pi \) of a configuration \( \sigma \) as above (i.e. such that \( \sigma(y) = 1 \) for all \( y \in \Lambda_N^c \)) is defined as the union of all the disagreement segments generated by \( \sigma \).

- A closed circuit in \( \pi \) is made by a sequence \( (\kappa_1, ..., \kappa_m) \) of “oriented” disagreement segments where \( \kappa_i = [r_i', r_i''], i = 1, ..., m, \) (all endpoints \( r_i', r_i'' \) in the dual lattice) and \( r_i'' = r_{i+1}' \) with \( r_{m+1}' = r_1' \): namely the last point of \( \kappa_{i-1} \) is attached to the initial point of \( \kappa_i \) and \( \kappa_m \) ends where \( \kappa_1 \) starts.

- A closed circuit \( \gamma = (\kappa_1, ..., \kappa_m) \) in \( \pi \) is a contour if (with the above notation) at all crossing points \( r_i'' \) where four disagreement segments meet then \((\kappa_i, \kappa_{i+1}) \) follows either the south-east or the north-west connection. Contours \( \gamma \) which only differ by their orientation are identified. Thus a contour \( \gamma \) is a closed circuit in \( \pi \) which may have self intersections but not self-crossings.
Repeating the above construction we decompose uniquely $\pi$ into a set $\gamma = (\gamma_1, \ldots, \gamma_n)$ of contours: any two contours $\gamma_i$ and $\gamma_j$ are either disjoint or have intersection at points in the dual lattice. If $r^*$ is such a point then $\gamma_i$ has a north-west and $\gamma_j$ a south-east turn at $r^*$ (or vice versa). $\gamma_i$ and $\gamma_j$ are then called compatible.

To visualize the construction of $(\gamma_1, \ldots, \gamma_n)$ it is convenient to “infinitesimally” smoothen $\pi$ at all self-crossing corners by the above south-east and north-west procedure. Then $\pi$ is made of disjoint closed curves $(c_1, \ldots, c_n)$ without self-intersections and their limits, when we remove the smoothing, are the contours $(\gamma_1, \ldots, \gamma_n)$.

Each $c_i$ separates $\mathbb{Z}^2$ into an interior and an exterior of $c_i$, denoted by int($\gamma_i$) and ext($\gamma_i$); the spins at the boundary of int($\gamma_i$) have all the same sign which is opposite to those at the boundary of ext($\gamma_i$). It then follows that $\sigma(x) = 1$ if $x$ is contained in an even number of contours and $\sigma(x) = -1$ otherwise.

The above allows to reconstruct $\sigma$ from $(\gamma_1, \ldots, \gamma_n)$ and it thus proves that there is a one to one correspondence between configurations $\sigma$ which are equal to $+1$ outside $\Lambda_N$ and configurations $(\gamma_1, \ldots, \gamma_n)$ of contours.

**Exercise.** Let $\sigma^\pm$ be such that $\sigma^\pm(x) = 1$ if $x = (x_1, x_2) \in \Lambda_N$ and $x_2 \geq 0$; $\sigma^\pm(x) = -1$ if $x \in \Lambda_N$ and $x_2 < 0$. Which are the possible contours of $\sigma^\pm$?

By the last property we can regard the Gibbs measure as a probability on contour configurations. We write the Ising interaction between two spins at an (unordered) pair $x \sim y$ of nearest neighbor sites as

$$-J\sigma(x)\sigma(y) = J1_{\sigma(x)\neq\sigma(y)} - J1_{\sigma(x)=\sigma(y)} = 2J1_{\sigma(x)\neq\sigma(y)} - J$$

Then the Ising hamiltonian (5.2) becomes

$$H_{\Lambda_N}(\sigma_{\Lambda_N} | \sigma_{\Lambda_N}^0 = +) = \sum_{\{x,y\} \cap \Lambda_N \neq \emptyset} 2J1_{\sigma(x)\neq\sigma(y)} + C$$

(10.3)

where the sum is over unordered pairs of nearest neighbor sites where at least one of them is in $\Lambda_N$. Namely each disagreement segment in $D_{N+\frac{1}{2}}$ contributes to the energy with a factor $2J$, namely it pays a penalty $2J$. Since it will be multiplied by $\beta$ this is a severe penalty (in the low temperature regime). We call

$$H_{\Lambda_N}(\gamma) = 2J|\gamma| = \sum_{\gamma \in \gamma} 2J|\gamma|$$

(10.4)

$|\gamma|$ the length of $\gamma$. Then

$$\mu_{\Lambda_N, \beta, h = 0, \sigma_{\Lambda_N}^0 = +}(\sigma) = \nu(\gamma) := Z_{\Lambda_N, \beta}^{-1} e^{-2\beta J|\gamma|}, \quad Z_{\Lambda_N, \beta} = \sum_{\gamma} e^{-2\beta J|\gamma|}$$

(10.5)
where the sum is over all $\gamma = (\gamma_1, ..., \gamma_n)$ of compatible contours in $D_{N \frac{1}{2}}$. We can then bound the probability in (10.1) by

$$\mu_{\Lambda, N, h=0, \sigma_{\Lambda_N}^c} = + \{\sigma(x) = 1\} \geq \nu \left[\text{no contour has } x \text{ in its interior}\right]$$

(10.6)

hence (proof left as an exercise)

$$\mu_{\Lambda, N, h=0, \sigma_{\Lambda_N}^c} = + \{\sigma(x) = 1\} \geq 1 - \nu \left[\text{there is } \gamma \text{ such that } \text{int}(\gamma) \ni x \right] \geq 1 - \sum_{\gamma: \text{int}(\gamma) \ni x} e^{-2\beta J|\gamma|}$$

(10.7)

If int$(\gamma) \ni x = (x_1, x_2)$ there are $\ell \leq |\gamma|$ and a vertical disagreement segment in $\gamma$ with center $z := (x_1 - \ell - \frac{1}{2}, x_2)$. Then if $x^*$ is in the dual lattice

$$\sum_{\text{int}(\gamma) \ni x} e^{-2\beta J|\gamma|} \leq \sum_{\gamma \ni x^*} |\gamma| e^{-2\beta J|\gamma|} \leq \sum_{\ell \geq 4} \ell \cdot 3^{\ell-1} e^{-2\beta J\ell}$$

(10.8)

The series is convergent if $2\beta J > \log 3$ and for all $\beta$ large enough becomes $< \frac{1}{2}$, hence (10.1).

11 \hspace{1em} \textbf{Cluster expansion at low temperatures.}

The analysis in Section 7 does not say how to compute the pressure, it just shows that it is well defined and independent of the boundary conditions. We will get here a rather explicit expression using contours and cluster expansion (at low temperatures and no magnetic field).

By (10.5) the partition function with plus or minus boundary conditions can be written as

$$Z_{\Lambda, N, \beta} = \sum_{\gamma \in \Gamma} \prod_{\gamma \subset \gamma} e^{-2\beta J|\gamma|}$$

(11.1)

where the sum is over all possible collections $\gamma$ of compatible contours. We can regard

$$Z_{\Lambda, N, \beta} = Z_{\Lambda, N, \beta}(z_1, ..., z_M)$$

(11.2)

where we have ordered in some arbitrary fashion the finite set (say of cardinality $M$) of all possible contours $\gamma$ and called

$$z_i := e^{-2\beta J|\gamma|} \text{ if } \gamma \text{ is the } i\text{-th contour}$$

(11.3)
$Z_{A_N,\beta}$ is then a polynomial function of the point $z = (z_1, \ldots, z_M) \in \mathbb{R}^M$ and its log can be expanded as a power series around $z = 0$ as $Z_{A_N,\beta}(0, \ldots, 0) = 1$. The Taylor series converges if $z$ is small enough and

$$\log Z_{A_N,\beta} = \sum_{n_1, \ldots, n_M} c_{n_1, \ldots, n_M} z_1^{n_1} \cdots z_M^{n_M} \quad (11.4)$$

(11.4) can be written in a compact way as

$$\log Z_{A_N,\beta} = \sum_{I} \alpha(I) z^I \quad (11.5)$$

where $I : \{1, \ldots, M\} \to \mathbb{N}_+$, i.e. $I(k) \in \{0, 1, \ldots\}$, $\alpha(I) = c_{I(1), \ldots, I(M)}$ and $Z^I = \prod z_k^{I(k)}$.

As remarked (11.4) holds for $z$ small enough but since we are interested in the thermodynamic limit where $M \to \infty$ the smallness condition is too strong to be useful. However suppose that

$$Z_{A_N,\beta} = \prod_{i=1}^M (1 - z_i) \quad (11.6)$$

then the condition for the expansion is that all $|z| < 1$ which can be verified even if $|z|$ is large. (11.6) can be relaxed because the contours have small weight so that the factorization property may not be so wrong. This has been made quantitative in the Kotecky-Preiss theory. The right hand side of (11.1) is called the polymer partition function and the quantity $w_\gamma := e^{-2\beta J|\gamma|}$ denotes the activity of the polymer $\gamma$.

**Lemma 11.1.** If $\beta$ is large enough for any finite contour $\gamma$

$$\sum_{\gamma' \{\text{no } \sim\} \gamma} w_{\gamma'} e^{\gamma'} \leq |\gamma| \quad (11.7)$$

where the sum is over all finite contours $\gamma'$ which are not compatible with $\gamma$: $\gamma' \{\text{no } \sim\} \gamma$ means that $\gamma'$ and $\gamma$ are not compatible. We will also say that $\gamma'$ and $\gamma$ are connected if they are not compatible.

**Exercise 11.1.** Prove that (11.7) is implied by: for any $x^*$ in the dual lattice

$$\sum_{\gamma \ni x^*} w_\gamma e^{\gamma} \leq 1 \quad (11.8)$$

**Proof of the lemma.** By the exercise it is enough to prove (11.8). Call $\ell$ the length of $\gamma$, then

$$\sum_{\gamma \ni x^*} w_\gamma e^{\gamma} \leq \sum_{\ell \geq 4} 4 \cdot 3^{\ell-1} e^{-2\beta J \ell} \quad (11.9)$$

If $\log 3 < 2\beta J$ the series converges exponentially and vanishes in the limit $\beta \to \infty$. \hfill \Box
The Kotecky-Preiss theory states that if (11.7) holds then \( Z_{\Lambda_N, \beta} \) can be written as an absolutely convergent series (i.e. the series of the absolute values is convergent):

\[
\log Z_{\Lambda_N, \beta} = \sum_{I \in \mathcal{I}_N} \alpha(I) \prod_{\gamma} w_{\gamma}^{I(\gamma)}
\]  

(11.10)

where \( \mathcal{I}_N \) is the space of all integer valued functions \( I = I(\gamma) \geq 0 \) on the space of all contours \( \gamma \) in \( D_N \) with the restriction that the set \( \{ \gamma : I(\gamma) \geq 1 \} \) should be connected, see Lemma 11.1 for the definition. \( \alpha(I) \) are (signed) combinatorial coefficients which are determined by the Taylor expansion of the log of the partition function, the beauty of the Kotecky-Preiss theory is that at least to some extent their precise value can be ignored. The theory says that for any \( \gamma \)

\[
\sum_{I : I(\gamma) \geq 1} |\alpha(I)| I(\gamma) \prod_{\gamma'} w_{\gamma'}^{I(\gamma')} \leq w_{\gamma} e^{|\gamma|}
\]  

(11.11)

We will use the following corollary of (11.11) (proof left as an exercise) to get a rather explicit expression for the thermodynamic pressure (and not merely its existence).

**Theorem 11.2.** Suppose that there is \( b > 0 \) so that for any \( x^* \) in the dual lattice

\[
\sum_{\gamma \ni x^*} w_{\gamma} e^{(1+b)|\gamma|} \leq 1
\]  

(11.12)

Then given any non negative function \( f(I) \) such that

\[
\sup_I \{f(I) e^{-b\|I\|}\} < \infty, \quad \|I\| = \sum_{\gamma} |\gamma| I(\gamma)
\]  

(11.13)

and any \( \gamma \) we have

\[
\sum_{I : I(\gamma) \geq 1} f(I) I(\gamma) |\alpha(I)| \prod_{\gamma'} w_{\gamma'}^{I(\gamma')} \leq w_{\gamma} e^{(1+b)|\gamma|} \sup_I \{f(I) e^{-b\|I\|}\}
\]  

(11.14)

Observe that given any \( b > 0 \) it is enough to choose \( \beta \) large enough so that (11.12) is satisfied.

The expression (11.10) shows that the the log of the partition function is “almost” an extensive function (which is true only in the thermodynamic limit). In fact we can write

\[
\log Z_{\Lambda_N, \beta} = \beta \sum_{x^* \in D_N} p_{x^*, \Lambda_N}, \quad \beta p_{x^*, \Lambda_N} := \sum_{I \in \mathcal{I}_N : I \ni x^*} c(I)^{-1} \alpha(I) \prod_{\gamma} w_{\gamma}^{I(\gamma)}
\]  

(11.15)

where \( x^* \) is in the dual lattice,

\[
c(I) := \left| \{y^* : y^* \in I\} \right|
\]  

(11.16)
I \ni x^* means that there is γ such that I(γ) ≥ 1 and x^* ∈ γ. To derive (11.15) we observe that the same I appears c(I) times in (11.15). We then define:

\[ p_{x^*} := \beta^{-1} \sum_{I : I \ni x^*} c(I)^{-1} \alpha(I) \prod_{\gamma} w_I^I(\gamma) \]  

(11.17)

(the series being absolutely convergent). (11.17) differs from (11.15) because the sum is over all I, i.e. without the restriction to be in \( \mathcal{I}_N \). By translation invariance

\[ p_{x^*} = P \quad \text{for any } x^* \]  

(11.18)

**Theorem 11.3.** \( P \) is equal to the thermodynamic pressure, i.e.

\[ P = \lim_{N \to \infty} \frac{1}{\beta |\Lambda_N|} \log Z_{\Lambda_N,\beta} \]  

(11.19)

**Proof.** Call \( D_{N+\frac{3}{2}} \) the points in the dual lattice which are in the boundary of \( \mathcal{D}_{N+\frac{3}{2}} \) (see the notation of the previous section), then

\[ \beta |p_{x^*,\Lambda_N} - p_{x^*}| \leq \sum_{y^* \in D_{N+\frac{3}{2}} \ni x^*, I \ni y^*} \sum_{I : I \ni x^*} |\alpha(I)| \prod_{\gamma} w_I^I(\gamma) \]  

(11.20)

Call

\[ |I| = \sum_{\gamma} |\gamma| 1_{I(\gamma) \geq 1} \]  

(11.21)

then

\[ |x^* - y^*| \leq |I| \quad \text{if } I \ni x^*, I \ni y^* \]  

(11.22)

and

\[ \beta |p_{x^*,\Lambda_N} - p_{x^*}| \leq \sum_{y^* \in D_{N+\frac{3}{2}} \ni x^*, I \ni y^*} \sum_{I : I \ni x^*} |\alpha(I)| \prod_{\gamma} w_I^I(\gamma) 1_{|I| \geq \|x^* - y^*\|} \]  

(11.23)

In order to apply Theorem 11.2 we rewrite the right hand side as

\[ \beta |p_{x^*,\Lambda_N} - p_{x^*}| \leq \sum_{y^* \in D_{N+\frac{3}{2}} \ni x^*, I : I(\gamma) \geq 1} \sum_{\gamma} |\alpha(I)| \prod_{\gamma} w_I^I(\gamma) 1_{|I| \geq \|x^* - y^*\|} \]  

(11.24)

By (11.14)

\[ \beta |p_{x^*,\Lambda_N} - p_{x^*}| \leq \sum_{y^* \in D_{N+\frac{3}{2}} \ni x^*} \sum_{\gamma \ni x^*} w_\gamma e^{(1+b)|\gamma|} e^{-b\|x^* - y^*\|} \]  

(11.25)

By (11.12)

\[ \beta |p_{x^*,\Lambda_N} - p_{x^*}| \leq \sum_{y^* \in D_{N+\frac{3}{2}}} e^{-b\|x^* - y^*\|} \]  

(11.26)
The proof is then completed by the exercise:

**Exercise 11.2.** *Prove that*

\[
\lim_{N \to \infty} \frac{1}{|\Lambda_N|} \sum_{x^* \in D_{N+\frac{1}{2}}} \sum_{y^* \in D_{N+\frac{3}{2}}} e^{-b|x^* - y^*|} = 0 \quad (11.27)
\]